Application of Homotopy Perturbation Method to Nonlinear Equations Describing Cocurrent and Countercurrent Imbibition in Fractured Porous Media

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Abstract

In oil industry, spontaneous imbibition is an important phenomenon in recovery from fractured reservoirs which can be defined as spontaneous uptake of a wetting fluid into a porous solid. Spontaneous imbibition involves both cocurrent and countercurrent flows. When a matrix block is partially covered by water, oil recovery is dominated by cocurrent imbibition i.e. the production of non wetting phase has the same direction of flow as the wetting phase. However if the matrix block is completely covered by water then countercurrent flow takes place, and the production of non wetting phase has an opposite direction of flow to that of the imbibing wetting phase. Each of these processes can be described by a nonlinear partial differential equation (PDE). In this paper, the homotopy perturbation method (HPM) which is a powerful series-based analytical tool, is used to approximate the solutions of cocurrent and countercurrent equations. HPM decomposes a complex partial differential equation under study to a series of simple ordinary differential equations that are easy to be solved. The solutions obtained by HPM are compared with that found using a common numerical method applied by MATLAB software. The difference between the two is seemed to be virtually negligible. A good agreement is also achieved from the comparison of the solutions obtained by HPM with those of a numerical method (NM).

Keywords: Fracture porous media, Homotopy perturbation method (HPM), Cocurrent imbibition, Countercurrent imbibition, Spontaneous imbibition

Introduction

Fractured petroleum reservoirs represent over 20% of the world's oil and gas reserves [1]. Fractured reservoirs are composed of interconnected pores (matrix system) and channels (fracture system). Usually matrix system contains most of the fluid in place but it has low permeability. The interconnected fracture system provides the main flow paths and a low storage volume. Most of the major naturally fractured reservoirs have active aquifers associated with them, or would eventually resort to some kind of secondary recovery process such as waterflooding [2]. During water injection or aquifer water movement, if capillary forces are strong, water imbibes as the wetting phase into the matrix blocks and discharges oil as non-wetting phase out of the block. Imbibition can occur in a reservoir in both countercurrent and cocurrent flow modes in proportions that depend on the ratio of gravity to capillary forces and on the conditions applied to the boundaries of the blocks. When a matrix block is partially covered by water, oil recovery is dominated by cocurrent imbibition, the production of non wetting phase has the same direction of flow as the wetting phase. However if the matrix block is completely covered by water then countercurrent flow takes place, and the production of non wetting phase has an opposite direction of flow to that of the imbibing wetting phase [3-7]. Figures 1, 2 show a simple schematic of co-and countercurrent imbibition, respectively. Cocurrent and countercurrent flows have been studied by many researchers [8-11]. Morrow and Mason [12] performed a review on recovery of oil by spontaneous imbibition. Kashchiev and Firoozabadi [13] presented analytical solutions for the initial

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stage of one-dimensional countercurrent flow of water and oil in porous media. Tavassoli et al. [14] studied countercurrent imbibition and used an approximate analytical approach to derive an expression for saturation profile. Silin and Patzek [15] extended a model of countercurrent imbibition based on Barenblatt's theory of non-equilibrium two-phase flow. Fine grid, one- and two-dimensional simulations of countercurrent imbibition were performed by Behbahani et al. [16].

To find the saturation distribution in matrix block in both co- and countercurrent flows, pertaining nonlinear PDEs should be solved. In solving these PDEs by numerical methods, stability and convergence should be taken into consideration. Otherwise, solutions might lead to inappropriate results. A semi exact method called Homotopy perturbation has been recently established and many authors have applied it to solve nonlinear equations [17-25].

The purpose of this study is to solve nonlinear PDEs describing co- and countercurrent imbibition by He's Homotopy perturbation method (HPM). The HPM was introduced by He [26-32]. In this method the solution is considered as the summation of an infinite series which usually converges rapidly to the exact solutions. Using homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0,1]$, which is considered a "small parameter"[33]. In addition, the equations are also solved by a numerical method and the results are then compared with those obtained by HPM. Moreover, in this study the effect of gravitational acceleration in both co- and countercurrent imbibition equations is considered as well.

2. Fundamentals of homotopy perturbation method (HPM)

To explain the basic idea of the HPM for solving nonlinear differential equations, we consider the following nonlinear differential equation:

$$A(w) - f(r) = 0, \quad (1)$$

Subject to boundary condition

$$\beta\left(w, \frac{\partial w}{\partial n}\right) = 0, \quad (2)$$

Where A is a general differential operator, β a boundary operator, $f(r)$ is a known analytical function, Γ is the boundary of domain, Ω and $\partial w / \partial n$ denotes differentiation along the normal drawn outwards from Ω. The operator A generally can be divided into two parts: a linear part M and a nonlinear part N. Equation 1 can therefore be rewritten as follows:

$$M(w) + N(w) - f(r) = 0, \quad (3)$$

In case the nonlinear Eq. 1 has no "small parameter", the following homotopy can be constructed,

$$H(v,P) = (1-P)[M(v) - M(w_0)] + P(M(v) + N(v) - f(r)) = 0, \quad (4)$$

Where P is called homotopy parameter and w_0 is an initial approximation which satisfies the boundary conditions. According to the HPM, the approximate solution of Eq. 4 can be expressed as a series of the power of P, i.e.

$$v = v_{0+} P v_1 + P^2 v_2 + \cdots, \quad (5a)$$

$$w = \lim_{p \to 1} v = v_0 + v_1 + v_{2+\cdots,} \quad (5b)$$

Where Eq. 4 corresponds to Eq. 1 and Eq. 5b becomes the approximate solution of Eq. 1.

Some interesting results have been attained using HPM [18, 19, 24, 34 and 35]. Ganji [25] compared Homotopy perturbation method (HPM) with numerical method in the heat transfer field. To do this, he considered two nonlinear PDEs related to two cases: (1) Cooling of a lumped system with variable specific heat, and (2) The temperature distribution equation in a thick rectangular fin radiation to free space. He showed that He's Homotopy perturbation method (HPM) overcomes completely the inaccurate results obtained by numerical methods especially in cases where the equation is intensively dependent on time. Ganji and Rajabi [24] attempted to show the capabilities and wide-range applications of the Homotopy perturbation method in comparison with the previous ones in solving heat transfer problems. In their research, Homotopy perturbation method was used to solve an unsteady nonlinear convective-radiative equation and a nonlinear convective-radiative conduction equation. They also showed that HPM has the smaller error compared to exact solution as the rate of nonlinearity is higher. Cveticanin [18] used the Homotopy perturbation method proposed by J.-H. He to solve pure strong nonlinear second-order differential equation. Two types of differential equations were considered: with strong cubic and strong quadratic nonlinearity. The obtained solution was compared with exact numerical one. The difference between these solutions was negligible for a long time period. The method was found to work extremely well in the examples. Siddiqui et al.[36] analyzed the thin film flow problem with a third grade fluid on an inclined plane. The governing non-linear equation was solved for the velocity field using the traditional perturbation technique as well as Homotopy perturbation method and the results were compared which were in complete agreement. Biazar and Ghazvini [37] presented the use of the He's Homotopy perturbation method, for systems of linear and non-linear Volterra integral equations of the second kind. For linear and non-linear systems, very good approximations were derived to the solutions. They concluded that the He's Homotopy perturbation method is a powerful and efficient technique in finding very good solutions for this kind of systems. Belendez et al. [21] used Homotopy perturbation method to solve the nonlinear differential equation that governs the nonlinear oscillations of a system typified as a mass attached to a stretched elastic wire. It was found that this perturbation method works very well for the whole range of parameters involved, and excellent agreement of the approximate frequencies and periodic solutions with the exact ones can be obtained. Cveticanin [38] used He's homotopy perturbation method to solve non-linear partial differential equations. An approximate solution of the differential equation which describes the longitudinal vibration of a beam was obtained. The solution was compared with that found using the variational iteration method introduced by He. The difference between the two solutions is negligible. Ravi Kanth and Aruna [39] tried to find the numerical solution of linear and non-linear higher-order boundary value problems using He's Homotopy perturbation method. This technique was tested on three examples, and was seen to produce satisfactory results. Cai and Wu [40] applied the Homotopy perturbation method to nonlinear oscillations. It was demonstrated that the solution procedure is of deceptively simplicity and the obtained insightful solutions are of high accuracy even with the first-order approximation. Fathizadeh and Rashidi [41] solved convective heat transfer equations of boundary layer with pressure gradient over a flat plate using Homotopy Perturbation Method (HPM). They showed that results agree well with those obtained numerically. Yildirim and Sezer [33] solved the steady two-dimensional laminar forced magnetohydrodynamic Hiemenz flow against a flat plate with variable wall temperature in a porous medium using the homotopy perturbation method (HPM). The skin friction coefficient and the rate of heat transfer given by the HPM were in good agreement with the numerical solutions of the Keller box method.

It should be pointed out that the following benefits has been suggested by different authors [25, 18, 21, 40 and 41]:

1. The HPM is valid for all the nonlinear equations with high order of nonlinearity containing different parameters.
2. The suggested method works well due to the fact that it uses the advantages of the homotopy, perturbation and power series expansion methods.

3. This technique yields a very rapid convergence of the solution series; in most cases only one iteration leads to high accuracy of the solution.
4. Due to the simplicity and accuracy of the He's homotopy perturbation method the application of this method is recommended when solving practical technical problems. Both the reliability of the method and the possibility of using computers, to obtain a more accurate solution, are the main reasons for wide applications of this method.

The reliability of the method and the reduction in the size of computational domain give this method a wider applicability.

3. Governing equations

Two phase immiscible flow of water (wetting phase) and oil (nonwetting phase) in a porous medium is governed by equations of conservation of mass and Darcy's law. In one dimensional (vertical) case and when flow is incompressible, mass conservation is expressed by:

$$\frac{\partial(\emptyset \rho_a S_a)}{\partial t} = -\frac{\partial(\rho_a u_a)}{\partial z} + q_a, \;\; a = w, o, \qquad (6)$$

Where φ is porosity (ratio of void space to grain volume) of porous medium, ρ is density of each fluid, S is saturation (fraction of pore space filled by a specific fluid) of each phase, u is Darcy velocity, q is water or oil mass flow rate which is zero in this study and z is depth. In addition, the index α denotes water/oil phase.

Also, Darcy's law has the following relation:

$$u_a = -\frac{K k_{ra}}{\mu_a}\frac{\partial}{\partial z}(p_a), \;\; a = w, o, \qquad (7)$$

Where p, μ, k_r are pressure, viscosity and relative permeability of each phase respectively. K is the absolute permeability of porous medium and g is gravitational acceleration.

It is noteworthy that absolute permeability is a measure of the ability of a porous medium to allow petroleum fluids to flow through its interconnected pores. Also, relative permeability demonstrates ability of one phase to flow in the presence of other phase(s), since the presence of more than one fluid generally inhibits flow.

In a porous medium, the following relation exists between water and oil saturations:

$$S_w + S_o = 1, \qquad (8)$$

The difference between pressures of oil and water is defined as capillary pressure:

$$P_c(S_w) = P_o - P_w, \qquad (9)$$

Also, the sum of Darcy velocities of water and oil phases is the total velocity:

$$u = u_w + u_o, \qquad (10)$$

Applying Eqs.9 and 10 to Eqs. 6 and 7 for water phase with some mathematical manipulation gives [42]

$$\frac{\partial(\varphi\, S_w)}{\partial t} + \frac{\partial}{\partial z}\{K f_w(S_w)\lambda_o(S_w)[\frac{dp_c}{dS_w}.\frac{dS_w}{dz} + (\rho_w - \rho_o)g] + f_w(S_w)\, u\} = 0 \qquad (11)$$

Where λ_o, λ_w and f_w are, oil and water mobility and water fractional flow function respectively, given by

$$\lambda_o = \frac{k_{ro}}{\mu_o}, \qquad (12)$$

$$\lambda_w = \frac{k_{rw}}{\mu_w}, \qquad (13)$$

$$f_w = \frac{\frac{k_{rw}}{\mu_w}}{\frac{k_{ro}}{\mu_o}+\frac{k_{rw}}{\mu_w}}, \qquad (14)$$

In Eq. 11, the value of u can be replaced by Eq. 15 in which total mobility and total pressure are used:

$$u = -K\left(\lambda(S)\frac{\partial P}{\partial z} - (\lambda_w \rho_{w+}\lambda_o \rho_o)g\right), \qquad (15)$$

Where

$$\lambda(S)\frac{\partial P}{\partial z} = \lambda_w \frac{\partial P_w}{\partial z} + \lambda_o \frac{\partial P_0}{\partial z}, \qquad (16)$$

By applying Eqs.8 and 10 to Eq. 6 and under the assumption that the fluids are incompressible,

$$\frac{\partial u}{\partial z} = 0, \qquad (17)$$

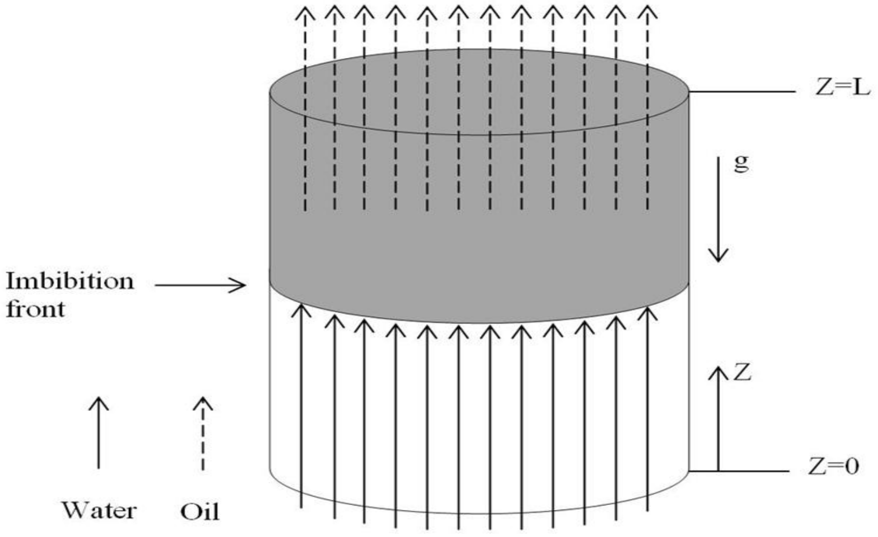

Figure 1: Simple schematic of 1D cocurrent flow into a vertical porous medium with gravity included

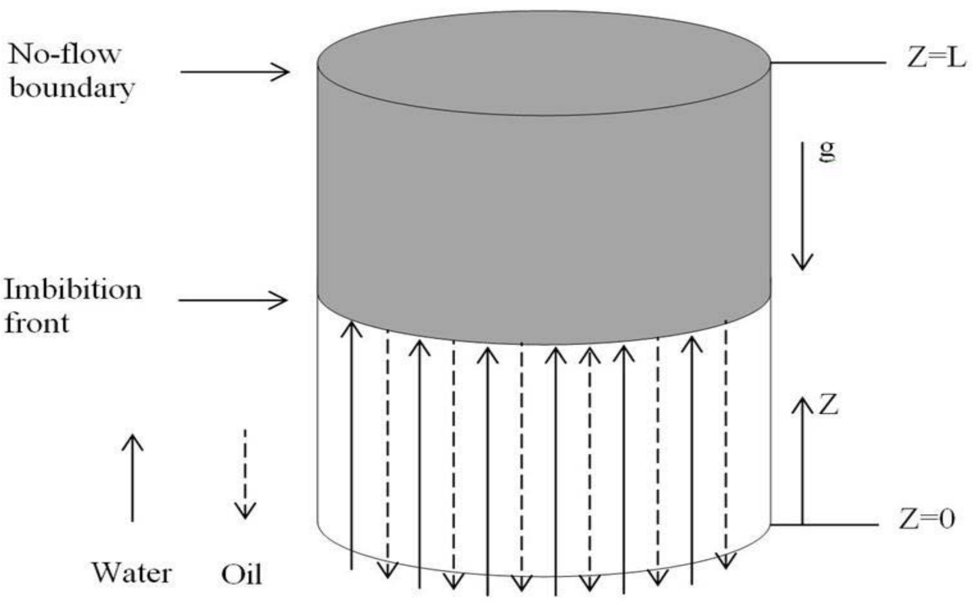

Figure 2: Simple schematic of 1D countercurrent flow into a vertical porous medium with gravity included

3.1 Cocurrent imbibition

By considering Eq. 17, Eq. 11 can be rewritten as

$$\frac{\partial(\varphi S_w)}{\partial t}+[u\frac{\partial f_w(S_w)}{\partial S_w}+\frac{\partial(f_w(S_w)\lambda_o(S_w))}{\partial S_w}K(\rho_w-\rho_o)g]\frac{\partial S_w}{\partial z}+$$
$$\frac{\partial}{\partial z}[Kf_w(S_w)\lambda_o(S_w)\frac{dp_c}{dS_w}.\frac{dS_w}{dz}]=0 \qquad (18)$$

The above equation can model cocurrent imbibition process. By using (15-17) and Eq. 18 saturation distribution of each phase can be derived as a function of time and vertical direction of matrix block. The following boundary and initial conditions are used for solving these equations:

$$S_w = e^{-z},\ \ t=0,\ \ 0\le\ z\le L, \qquad (19)$$
$$S_w = 1,\ \ t>0,\ \ z=0, \qquad (20)$$
$$S_w = e^{-L},\ \ t>0,\ \ z=L, \qquad (21)$$
$$P_w = 0,\ \ t<0,\ \ z=0, \qquad (22)$$
$$P_o = 0,\ \ t<0,\ \ z=L, \qquad (23)$$

3.2 Countercurrent imbibition

In countercurrent imbibition process, total velocity is zero. So, Eq. 18 becomes

$$\frac{\partial(\varphi S_w)}{\partial t}+[\frac{\partial(f_w(S_w)\lambda_o(S_w))}{\partial S_w}K(\rho_w-\rho_o)g]\frac{\partial S_w}{\partial z}+$$
$$\frac{\partial}{\partial z}[Kf_w(S_w)\lambda_o(S_w)\frac{dp_c}{dS_w}.\frac{dS_w}{dz}] \qquad (24)$$

Again, using (15-17) and Eq. 24, saturation profile can be found. In this case, initial and boundary conditions are

$$S_w = e^{-z},\ \ t=0,\ \ 0\le z\le L, \qquad (25)$$
$$S_w = 1,\ \ t>0,\ \ z=0, \qquad (26)$$
$$S_w = e^{-L},\ \ t>0,\ \ z=L, \qquad (27)$$

Where L is length of the matrix block.

4. Results and discussions

In this section, we apply HPM for solving PDEs derived for cocurrent and countercurrent imbibition processes in order to find saturation distribution in these two cases. For ease of mathematical calculation, we assume standard forms of Scheidegger and Johnson [43] for the analytical relationship between the relative permeability, phase saturation and capillary pressure as

$$k_{rw} = S_w, \qquad (28)$$
$$k_{ro} = 1 - aS_w, \qquad (29)$$
$$P_c = -BS_w, \qquad (30)$$

Where a and B are empirical constants that can be evaluated from measured data.

4.1 HMP solution of cocurrent imbibition equation

In this section, after some mathematical manipulations, we construct a homotopy of nonlinear partial differential system of Eq.17 and Eq. 18 as follows:

$$(1-P)[\mu_w^2\varphi\frac{\partial S_w}{\partial t}-\mu_w^2\varphi\frac{\partial S_{w0}}{\partial t}]+P\{(\mu_w+(\mu_o-a\mu_w)S_w)^2\frac{\partial(\varphi S_w)}{\partial t}+$$
$$[-K(\lambda_w\frac{\partial p_w}{\partial z}+\lambda_o\frac{\partial p_o}{\partial z}-(\lambda_w\rho_w+\lambda_o\rho_o)g)\mu_o\mu_w+$$
$$(\mu_w-2a\mu_w S_w+a^2\mu_w S_w^2-\mu_o a S_w^2)K(\rho_w-\rho_o)g]\frac{\partial S_w}{\partial z}+$$
$$K(-B)(\mu_w-2a\mu_w S_w+a^2\mu_w S_w^2-\mu_o a S_w^2)(\frac{\partial S_w}{\partial z})^2$$
$$+K(-B)(S_w-aS_w^2)(\mu_w+(\mu_o-a\mu_w)S_w)\frac{\partial^2 S_w}{\partial z^2}\}=0,$$
$$\frac{\partial^2 p_w}{\partial z^2}+P\frac{\mu_o-a\mu_w}{\mu_w}S_w\frac{\partial^2 p_w}{\partial z^2}+P\frac{\mu_o-a\mu_w}{\mu_w}\frac{\partial p_w}{\partial z}\frac{\partial S_w}{\partial z}-PB\frac{\partial^2 S_w}{\partial z^2}$$
$$PaBS_w\frac{\partial^2 S_w}{\partial z^2}+PB(\frac{\partial S_w}{\partial z})^2-P\frac{g(\mu_o\rho_w-a\mu_w\rho_o)}{\mu_w}\frac{\partial S_w}{\partial z}=0 \qquad (31)$$

We consider the approximate solutions as follows:

$$S_w = S_{w0}+PS_{w1}+P^2S_{w2}+\cdots, \qquad (32)$$
$$p_w = p_{w0}+Pp_{w1}+P^2p_{w2}+\cdots, \qquad (33)$$

Assuming $\frac{\partial S_{w0}}{\partial t}=\frac{\partial^2 p_{w0}}{\partial z^2}=0$ and substituting S_w and p_w from Eq. 32 and Eq. 33 into Eq. 31 and some simplification and rearranging based on powers of P-terms, we have:

$$P^0: \frac{\partial S_{w0}}{\partial t}=0, \frac{\partial^2 P_{w0}}{\partial z^2}=0, \qquad (34)$$

$$S_{w0}(z,0)=e^{-z},\ p_{w0}(0,t)=0,\ p_{o0}(0,t) \qquad (35)$$

$$P^1:\mu_w{}^2\frac{\partial S_{w1}}{\partial t}+(\frac{K(\rho_w-\rho_o)g\,\mu_w}{\varphi}+\frac{g\,\mu_w K\,\rho_o}{\varphi})\frac{\partial S_{w0}}{\partial z}+$$
$$(\frac{K(\rho_w-\rho_o)g\,\mu_w a^2}{\varphi}-\frac{K(\rho_w-\rho_o)g\,\mu_o a}{\varphi})S_{w0}{}^2\frac{\partial S_{w0}}{\partial z}-$$
$$\frac{KB\,\mu_w}{\varphi}S_{w0}\frac{\partial^2 S_{w0}}{\partial z^2}-\frac{KB(\mu_o-2a\mu_w)}{\varphi}S_{w0}{}^2\frac{\partial^2 S_{w0}}{\partial z^2}-$$
$$\frac{KB(a^2\mu_w-a\mu_o)}{\varphi}S_{w0}{}^3\frac{\partial^2 S_{w0}}{\partial z^2}+(\frac{gK(\mu_o\rho_w-a\mu_w\rho_o)}{\varphi}-$$
$$\frac{2K(\rho_w-\rho_o)g\,\mu_w a}{\varphi})S_{w0}\frac{\partial S_{w0}}{\partial z}+\frac{a\mu_w KB}{\varphi}S_{w0}(\frac{\partial S_{w0}}{\partial z})^2+$$
$$\frac{KBa(\mu_o-a\mu_w)}{\varphi}S_{w0}{}^2(\frac{\partial S_{w0}}{\partial z})^2-\frac{K\,\mu_w}{\varphi}.\frac{\partial S_{w0}}{\partial z}.\frac{\partial p_{w0}}{\partial z}-$$
$$\frac{K(\mu_o-a\mu_w)}{\varphi}S_{w0}\frac{\partial S_{w0}}{\partial z}.\frac{\partial p_{w0}}{\partial z}=0,$$
$$\frac{\partial^2 p_{w1}}{\partial z^2}+\frac{\mu_o-a\mu_w}{\mu_w}S_{w0}\frac{\partial^2 p_{w0}}{\partial z^2}+\frac{\mu_o-a\mu_w}{\mu_w}.\frac{\partial S_{w0}}{\partial z}.\frac{\partial p_{w0}}{\partial z}-$$
$$B\frac{\partial^2 S_{w0}}{\partial z^2}+aBS_{w0}\frac{\partial^2 S_{w0}}{\partial z^2}+B(\frac{\partial S_{w0}}{\partial z})^2-$$
$$\frac{g(\mu_o\rho_w-a\mu_w\rho_o)}{\mu_w}.\frac{\partial S_{w0}}{\partial z}=0 \tag{36}$$

$$S_{w1}(z,0)=0,p_{w1}(0,t)=0,p_{o1}(0,t)=0, \tag{37}$$

$$P^2:\mu_w{}^2\frac{\partial S_{w2}}{\partial t}+2\mu_w(\mu_o-a\mu_w)S_{w0}\frac{\partial S_{w1}}{\partial t}+$$
$$(\mu_o-a\mu_w)^2S_{w0}{}^2\frac{\partial S_{w1}}{\partial t}+(\frac{K(\rho_w-\rho_o)g\,\mu_w}{\varphi}+$$
$$\frac{K\rho_o g\,\mu_w}{\varphi})\frac{\partial S_{w1}}{\partial z}+(\frac{K(\rho_w-\rho_o)g\,\mu_w a^2}{\varphi}-$$
$$\frac{K(\rho_w-\rho_o)g\,\mu_o a}{\varphi})(2S_{w0}S_{w1}\frac{\partial S_{w0}}{\partial z}+S_{w0}{}^2\frac{\partial S_{w1}}{\partial z})+$$
$$(\frac{Kg(\mu_o\rho_w-a\mu_w\rho_o)}{\varphi}-\frac{2K(\rho_w-\rho_o)g\,\mu_w a}{\varphi})$$
$$(S_{w0}\frac{\partial S_{w1}}{\partial z}+S_{w1}\frac{\partial S_{w0}}{\partial z})-\frac{KB\,\mu_w}{\varphi}(S_{w0}\frac{\partial^2 S_{w1}}{\partial z^2}+$$
$$S_{w1}\frac{\partial^2 S_{w0}}{\partial z^2})-\frac{KB(\mu_o-2a\mu_w)}{\varphi}(2S_{w0}S_{w1}\frac{\partial^2 S_{w0}}{\partial z^2}+$$
$$S_{w0}{}^2\frac{\partial^2 S_{w1}}{\partial z^2})-\frac{2KB\,\mu_w}{\varphi}.\frac{\partial S_{w0}}{\partial z}.\frac{\partial S_{w1}}{\partial z}-\frac{KB(a^2\mu_w-a\mu_o)}{\varphi}$$
$$(3S_{w0}{}^2S_{w1}\frac{\partial^2 S_{w0}}{\partial z^2}+S_{w0}{}^3\frac{\partial^2 S_{w1}}{\partial z^2})+\frac{aBK\,\mu_w}{\varphi}(2S_{w0}$$
$$\frac{\partial S_{w0}}{\partial z}.\frac{\partial S_{w1}}{\partial z}+S_{w1}(\frac{\partial S_{w0}}{\partial z})^2)+\frac{KBa(\mu_o-a\mu_w)}{\varphi}$$
$$(2S_{w0}{}^2\frac{\partial S_{w0}}{\partial z}.\frac{\partial S_{w1}}{\partial z}+2S_{w0}S_{w1}(\frac{\partial S_{w0}}{\partial z})^2)-\frac{K\,\mu_w}{\varphi}$$
$$\frac{\partial S_{w0}}{\partial z}.\frac{\partial p_{w1}}{\partial z}-\frac{K(\mu_o-a\mu_w)}{\varphi}S_{w0}\frac{\partial S_{w0}}{\partial z}.\frac{\partial p_{w1}}{\partial z}+$$
$$2\frac{BK\,\mu_w}{\varphi}.\frac{\partial S_{w0}}{\partial z}.\frac{\partial S_{w1}}{\partial z}=0,$$

$$\frac{\partial^2 p_{w2}}{\partial z^2}+\frac{\mu_o-a\mu_w}{\mu_w}(S_{w0}\frac{\partial^2 p_{w1}}{\partial z^2}+S_{w1}\frac{\partial^2 p_{w0}}{\partial z^2})+$$
$$\frac{\mu_o-a\mu_w}{\mu_w}(\frac{\partial S_{w0}}{\partial z}.\frac{\partial p_{w1}}{\partial z}+\frac{\partial S_{w1}}{\partial z}.\frac{\partial p_{w0}}{\partial z})-B\frac{\partial^2 S_{w1}}{\partial z^2}$$
$$+aB(S_{w0}\frac{\partial^2 S_{w1}}{\partial z^2}+S_{w1}\frac{\partial^2 S_{w0}}{\partial z^2})+2B\frac{\partial S_{w0}}{\partial z}.\frac{\partial S_{w1}}{\partial z}-$$
$$\frac{g(\mu_o\rho_w-a\mu_w\rho_o)}{\mu_w}.\frac{\partial S_{w1}}{\partial z}=0, \tag{38}$$

$$S_{w2}(z,0)=0,p_{w2}(0,t)=0,p_{o2}(0,t)=0, \tag{39}$$

Solving (34-38) and applying boundary conditions (35-39) with data in Table 1, we have:

$$S_{w0}=e^{-z} \tag{40}$$
$$S_{w1}=[(3.821\times10^{-6})e^{-z}+(-3.914\times10^{-7})e^{-2z}+$$
$$(-9.237\times10^{-7})e^{-3z}+(-1.112\times10^{-7})e^{-4z}]t \tag{41}$$

$$S_{w2}=-[(-3.448\times10^{-7})e^{-z}+(2.483\times10^{-6})e^{-2z}+$$
$$(5.472\times10^{-7})e^{-3z}+(-2.135\times10^{-7})e^{-4z}+$$
$$(-4.923\times10^{-8})e^{-5z}+(-2.179\times10^{-9})e^{-6z}]t-$$
$$0.5t^2[(-1.460\times10^{-11})e^{-z}+(1.070\times10^{-6})e^{-2z}+$$
$$(1.227\times10^{-8})e^{-3z}+(2.061\times10^{-12})e^{-4z}+$$
$$(-6.945\times10^{-12})e^{-5z}+(-2.238\times10^{-12})e^{-6z}+$$
$$(-1.792\times10^{-13})e^{-7z}] \tag{42}$$

According to Eq. 32 and assumption that $P=1$, we get the following approximation for saturation distribution:

$$S_w(z,t)=e^{-z}+t[(3.821\times10^{-6})e^{-z}+(-3.914\times10^{-7})e^{-2z}+$$
$$(-9.237\times10^{-7})e^{-3z}+(-1.112\times10^{-7})e^{-4z}+$$
$$(3.448\times10^{-7})e^{-z}+(-2.483\times10^{-6})e^{-2z}+$$
$$(-5.472\times10^{-7})e^{-3z}+(2.135\times10^{-7})e^{-4z}+$$
$$(4.923\times10^{-8})e^{-5z}+(2.179\times10^{-9})e^{-6z}]-$$
$$0.5t^2[(-1.460\times10^{-11})e^{-z}+(1.070\times10^{-6})e^{-2z}+$$
$$(1.227\times10^{-8})e^{-3z}+(2.061\times10^{-12})e^{-4z}+$$
$$(-6.945\times10^{-12})e^{-5z}+(-2.238\times10^{-12})e^{-6z}+$$
$$(-1.792\times10^{-13})e^{-7z}] \tag{43}$$

4.2 HMP solution of countercurrent imbibition equation

We construct the homotopy of Eq. 24 which satisfies

$$(1-P)[\mu_w^{\,2}\varphi\frac{\partial S_w}{\partial t}-\mu_w^{\,2}\varphi\frac{\partial S_{w0}}{\partial t}]+$$
$$P\{(\mu_w+(\mu_o-a\mu_w)S_w)^2\frac{\partial(\varphi S_w)}{\partial t}+$$
$$[(\mu_w-2a\mu_w S_w+a^2\mu_w S_w^{\,2}-\mu_o aS_w^{\,2})K(\rho_w-\rho_o)g]\frac{\partial S_w}{\partial z}$$
$$+K(-B)(\mu_w-2a\mu_w S_w+a^2\mu_w S_w^{\,2}-\mu_o aS_w^{\,2})(\frac{\partial S_w}{\partial z})^2$$
$$+K(-B)(S_w-aS_w^{\,2})(\mu_w+(\mu_o-a\mu_w)S_w)\frac{\partial^2 S_w}{\partial z^2}\}=0 \qquad (44)$$

With initial approximation $\frac{\partial S_{w0}}{\partial t}=0$ and we suppose that the solution of Eq. 24 has the form:

$$S_w=S_{w0}+PS_{w1}+P^2S_{w2}+\cdots, \qquad (45)$$

Then, substituting Eq. 45 into Eq. 44, and equating the terms with identical powers of P,

$$P^0:\frac{\partial S_{w0}}{\partial t}=0, \qquad (46)$$

$$S_{w0}(z,0)=e^{-z}, \qquad (47)$$

$$P^1:\mu_w^{\,2}\frac{\partial S_{w1}}{\partial t}+(\frac{K(\rho_w-\rho_o)g\mu_w}{\varphi}+\frac{g\mu_w K\rho_o}{\varphi})\frac{\partial S_{w0}}{\partial z}+$$
$$(\frac{K(\rho_w-\rho_o)g\mu_w a^2}{\varphi}-\frac{K(\rho_w-\rho_o)g\mu_o a}{\varphi})S_{w0}^{\,2}\frac{\partial S_{w0}}{\partial z}-$$
$$\frac{KB\mu_w}{\varphi}S_{w0}\frac{\partial^2 S_{w0}}{\partial z^2}-\frac{KB(\mu_o-2a\mu_w)}{\varphi}S_{w0}^{\,2}\frac{\partial^2 S_{w0}}{\partial z^2}-$$
$$\frac{KB(a^2\mu_w-a\mu_o)}{\varphi}S_{w0}^{\,3}\frac{\partial^2 S_{w0}}{\partial z^2}+\frac{a\mu_w KB}{\varphi}S_{w0}(\frac{\partial S_{w0}}{\partial z})^2+$$
$$(\frac{gK(\mu_o\rho_w-a\mu_w\rho_o)}{\varphi}-\frac{2K(\rho_w-\rho_o)g\mu_w a}{\varphi})S_{w0}\frac{\partial S_{w0}}{\partial z}+$$
$$\frac{KBa(\mu_o-a\mu_w)}{\varphi}S_{w0}^{\,2}(\frac{\partial S_{w0}}{\partial z})^2=0, \qquad (48)$$

$$S_{w1}(z,0)=0, \qquad (49)$$

$$P^2:\mu_w^{\,2}\frac{\partial S_{w2}}{\partial t}+2\mu_w(\mu_o-a\mu_w)S_{w0}\frac{\partial S_{w1}}{\partial t}+$$
$$(\mu_o-a\mu_w)^2S_{w0}^{\,2}\frac{\partial S_{w1}}{\partial t}+(\frac{K(\rho_w-\rho_o)g\mu_w}{\varphi})\frac{\partial S_{w1}}{\partial z}+$$
$$(\frac{K(\rho_w-\rho_o)g\mu_w a^2}{\varphi}-\frac{K(\rho_w-\rho_o)g\mu_o a}{\varphi})(2S_{w0}S_{w1}\frac{\partial S_{w0}}{\partial z}+$$
$$S_{w0}^{\,2}\frac{\partial S_{w1}}{\partial z})-\frac{2K(\rho_w-\rho_o)g\mu_w a}{\varphi}(S_{w0}\frac{\partial S_{w1}}{\partial z}+S_{w1}\frac{\partial S_{w0}}{\partial z})-$$
$$\frac{KB\mu_w}{\varphi}(S_{w0}\frac{\partial^2 S_{w1}}{\partial z^2}+S_{w1}\frac{\partial^2 S_{w0}}{\partial z^2})-\frac{KB(\mu_o-2a\mu_w)}{\varphi}$$
$$(2S_{w0}S_{w1}\frac{\partial^2 S_{w0}}{\partial z^2}+S_{w0}^{\,2}\frac{\partial^2 S_{w1}}{\partial z^2})-\frac{2KB\mu_w}{\varphi}.\frac{\partial S_{w0}}{\partial z}.\frac{\partial S_{w1}}{\partial z}+$$
$$\frac{2aBK\mu_w}{\varphi}(2S_{w0}\frac{\partial S_{w0}}{\partial z}.\frac{\partial S_{w1}}{\partial z}+S_{w1}(\frac{\partial S_{w0}}{\partial z})^2)-$$
$$\frac{KB(a^2\mu_w-a\mu_o)}{\varphi}(3S_{w0}^{\,2}S_{w1}\frac{\partial^2 S_{w0}}{\partial z^2}+S_{w0}^{\,3}\frac{\partial^2 S_{w1}}{\partial z^2})+$$
$$\frac{KBa(\mu_o-a\mu_w)}{\varphi}(2S_{w0}^{\,2}\frac{\partial S_{w0}}{\partial z}.\frac{\partial S_{w1}}{\partial z}+$$
$$2S_{w0}S_{w1}(\frac{\partial S_{w0}}{\partial z})^2)=0, \qquad (50)$$

$$S_{w2}(z,0)=0, \qquad (51)$$

Solving (46-50) and applying boundary conditions (47-51) with data in Table 1, we have:

$$S_{w0}=e^{-z}, \qquad (52)$$
$$S_{w1}=[(1.157\times10^{-6})e^{-z}+(-1.853\times10^{-6})e^{-2z}+$$
$$(-1.321\times10^{-6})e^{-3z}+(-1.112\times10^{-7})e^{-4z}]t \qquad (53)$$

$$S_{w2}=-[(-3.239\times10^{-7})e^{-2z}+(4.960\times10^{-7})e^{-3z}+$$
$$(4.061\times10^{-7})e^{-4z}+(5.701\times10^{-8})e^{-5z}+$$
$$(2.179\times10^{-9})e^{-6z}]t-0.5t^2[(1.338\times10^{-12})e^{-z}+$$
$$(-8.571\times10^{-12})e^{-2z}+(-8.581\times10^{-13})e^{-3z}+$$
$$(1.752\times10^{-11})e^{-4z}+(1.534\times10^{-11})e^{-5z}+$$
$$(3.076\times10^{-12})e^{-6z}+(1.792\times10^{-13})e^{-7z}] \qquad (54)$$

According to Eq. 32 and assumption that $P=1$, we get the following approximation for saturation distribution:

$$\begin{aligned}S_w(z,t) = e^{-z} &+ [(1.157\times10^{-6})e^{-z} + (-1.853\times10^{-6})e^{-2z} + (-1.321\times10^{-6})e^{-3z} + (-1.112\times10^{-7})e^{-4z} \\ &]t - [(-3.239\times10^{-7})e^{-2z} + (4.960\times10^{-7})e^{-3z} + (4.061\times10^{-7})e^{-4z} + (5.701\times10^{-8})e^{-5z} + \\ &(2.179\times10^{-9})e^{-6z}]t - 0.5t^2[(1.338\times10^{-12})e^{-z} + (-8.571\times10^{-12})e^{-2z} + (-8.581\times10^{-13})e^{-3z} \\ &+ (1.752\times10^{-11})e^{-4z} + (1.534\times10^{-11})e^{-5z} + (3.076\times10^{-12})e^{-6z} + (1.792\times10^{-13})e^{-7z}]\end{aligned} \tag{55}$$

Table 1: Data for numerical calculations

K	$10^{-13} m^2$	μ_w	1.2 mPas
φ	0.23	μ_o	1.5 mPas
L	1.5 m	ρ_w	1090 $\mathrm{Kg/m^3}$
a	1.11	ρ_o	760 $\mathrm{Kg/m^3}$
B	1000 Pa		

Since Eqs. 17, 18, 24 cannot be easily solved by analytical methods, these equations are therefore, solved by a numerical method applying software MATLAB using function *pdepe.* Values of saturation of water obtained by numerical method and HPM, for a specific time and at different locations (z) in the matrix block, are given in Tables 2 and 3 for both co- and countercurrent imbibition processes, respectively. As can be seen, HPM has a high accuracy at $t = 2.5$ minutes and $t = 16$ hours for both co- and counter current imbibition processes.

Additionally, the consequent results of the two different methods of homotopy perturbation and numerical, for co-and countercurrent imbibition processes, are compared in Figs. 3,4,5,6 at some other times of the processes. Figures 4, 6 which are related to the saturation distribution in the block at early time, indicate a good agreement between the methods.

As mentioned before, HPM gives an approximation to the solution of differential equations. Usually, including more terms in the power series of P, gives higher accuracy in HPM. The reason underlying the less accurate solutions in the cases of t=24 hours (Fig.3) and t=5 minutes (Fig.5) compared to previous cases, is that the series are truncated after P^2-term. This problem can be solved by considering more terms in series of the power of P.

Table 2: The results of HPM and NM methods for saturation in cocurrent imbibition at $t = 2.5$ minutes

$z(m)$	S_w	
	HPM	NM
0.30	0.747999	0.743939
0.35	0.711204	0.713777
0.40	0.676232	0.680106
0.45	0.642994	0.647461
0.50	0.611401	0.616133
0.55	0.581371	0.586543
0.60	0.552826	0.553426
0.65	0.525691	0.530115
0.70	0.499896	0.504166
0.75	0.475374	0.479559
0.80	0.452062	0.452830
0.85	0.429898	0.436189
0.90	0.408827	0.412014
0.95	0.388793	0.391535
1.00	0.369746	0.370497
1.05	0.351635	0.358740
1.10	0.334415	0.336500
1.15	0.318042	0.319623
1.20	0.302473	0.303147
1.25	0.287669	0.295000
1.30	0.273592	0.274831
1.35	0.260206	0.261020
1.40	0.247477	0.248072
1.45	0.235373	0.242612
1.50	0.223862	0.223130

Table 3: The results of HPM and NM methods for saturation in countercurrent imbibition at t =16 hours

$z(m)$	S_w	
	HPM	NM
0.30	0.716510	0.711008
0.35	0.683785	0.679903
0.40	0.652695	0.650005
0.45	0.623099	0.621280
0.50	0.594883	0.593693
0.55	0.567951	0.567210
0.60	0.542223	0.541797
0.65	0.517629	0.517421
0.70	0.494111	0.494048
0.75	0.471614	0.471645
0.80	0.450092	0.450179
0.85	0.429501	0.429619
0.90	0.409801	0.409932
0.95	0.390956	0.391088
1.00	0.372930	0.373056
1.05	0.355691	0.355806
1.10	0.339207	0.339308
1.15	0.323448	0.323534
1.20	0.308385	0.308456
1.25	0.293990	0.294043
1.30	0.280236	0.280255
1.35	0.267098	0.267009
1.40	0.254550	0.254035
1.45	0.242568	0.240408
1.50	0.231129	0.223130

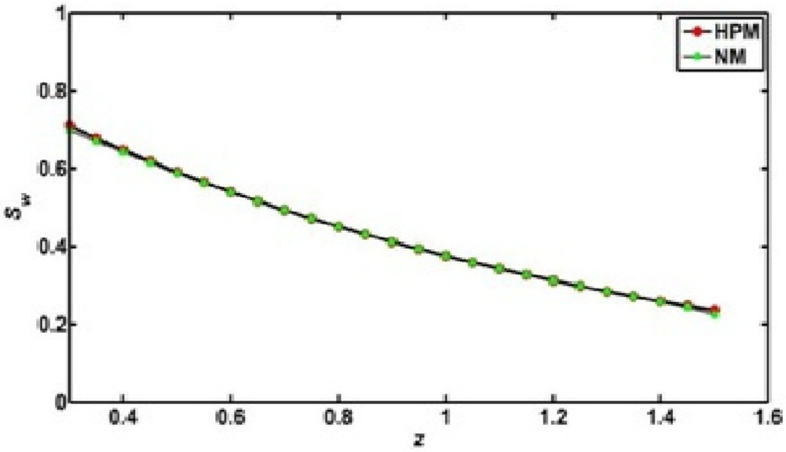

Figure 3: The comparison of the answers obtained by HPM and numerical methods for water saturation for counter current flow at t=24 hours

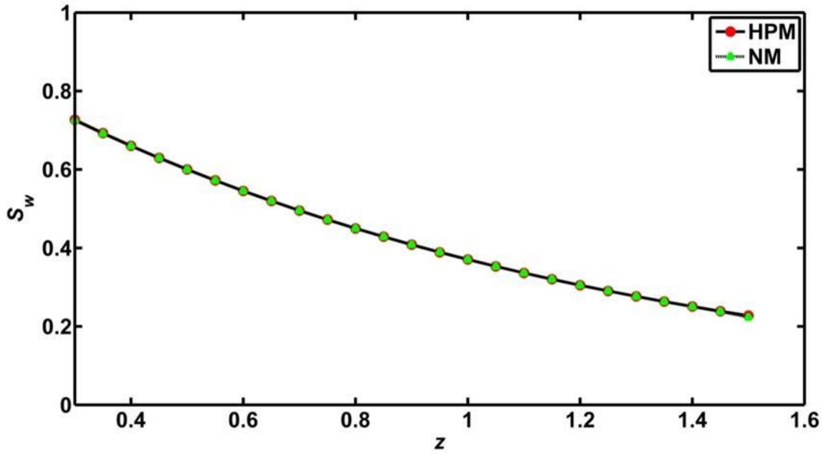

Figure 4: The comparison of the answers obtained by HPM and numerical methods for water saturation for counter current flow at t=8 hours

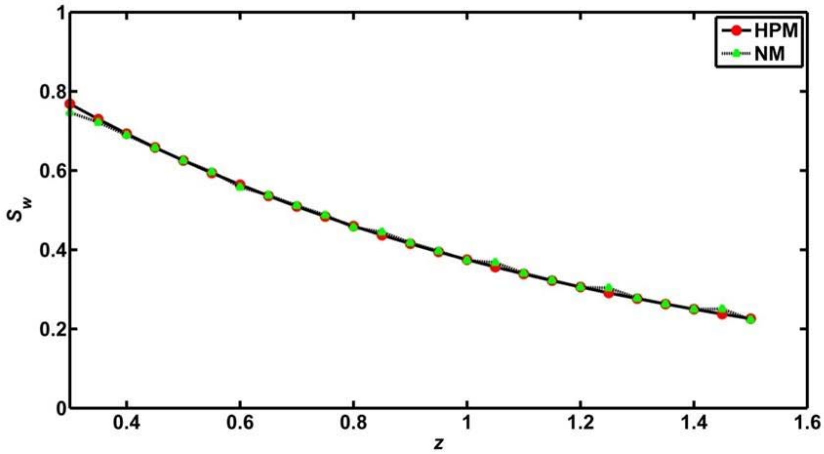

Figure 5: The comparison of the answers obtained by HPM and numerical methods for water saturation for cocurrent flow at t=5 minutes

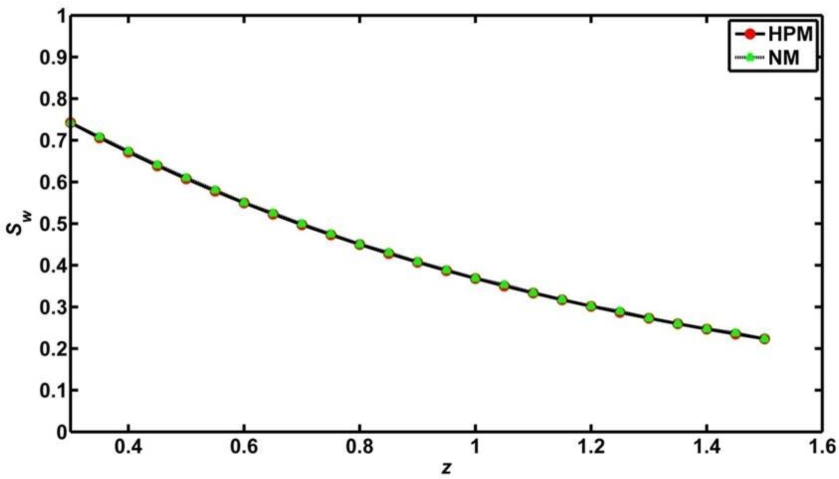

Figure 6: The comparison of the answers obtained by HPM and numerical methods for water saturation for cocurrent flow at t=1 minute

5. Conclusion

In this study, homotopy perturbation method is used for solving the problem of co-and counter current imbibition of water into a single matrix block to find the water saturation in a matrix block when two phase flow of oil and water exists while all parameters including gravitational acceleration are considered in the governing equations. The homotopy perturbation method is effective in solving strong non-linear partial differential equations. The strong non-linear partial differential equations are transformed into a system of ordinary differential equations which are suitable for calculation. The partial differential equations are solved by a numerical method applying MATLAB using function *pdepe*. The solutions which are obtained in this way are in good agreement with the solutions obtained by HPM.

The results show that HPM has high accuracy at early times of the co-and counter current imbibition processes. This accuracy can be high in middle and late time regions, as well, by including more terms in the series of the power of P, in procedure of HPM.

He's homotopy perturbation method can be a reliable tool for solving partial differential equations describing different engineering problems. Both the reliability of the method and the possibility of using computers, to obtain a more accurate solution, allows the method to be widely applied.

Abbreviations

A	General differential operator
a	Empirical constant
B	Empirical constant
$f(r)$	Known analytical function
f_w	Water fractional flow function
g	Gravitational acceleration
K	Absolute permeability
$k_{r\alpha}$	Relative permeability of each phase ($\alpha =$ oil,water)
L	Length of matrix block
M	Linear part of operator A
N	Nonlinear part of operator A
p_c	Capillary pressure
P	Homotopy parameter
p_{wj}(j=1, 2, 3,…)	Series terms of Eq.32
p_α	Pressure of each phase ($\alpha =$ oil, water)
S_{wj}	Series terms of Eq.32
S_α	Saturation of each phase ($\alpha =$ oil, water)
t	Time
U	Total velocity

ν	Solution of Eq. 4
w	Solution of Eq. 1
w_0	Initial approximation to w satisfying boundary conditions.
z	Depth
$\partial w/\partial n$	Differentiation along normal drawn Outwards from Ω

Greek symbols

β	Boundary operator
Γ	Boundary of domain Ω
Λ	Total mobility
λ_o	Oil mobility
λ_w	Water mobility
μ_α	Viscosity of each phase ($\alpha =$ oil, water)
ρ_α	Density of each phase ($\alpha =$ oil, water)
φ	porosity
Ω	Domain

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